RPA equations and the instantaneous Bethe-Salpeter equation

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We give a derivation of the particle-hole RPA equations for an interacting multi-fermion system by applying the instantaneous approximation to the amputated two-fermion propagator of the system. In relativistic field theory the same approximation leads from the fermion-antifermion Bethe-Salpeter equation to the Salpeter equation. We show that RPA equations and Salpeter equation are indeed equivalent.

I. INTRODUCTION

In the study of systems of many interacting fermions one finds that methods used in relativistic field theory usually correspond to approximations well-known in nonrelativistic many-particle theory. An interesting example is the correspondence of the Hartree-Fock approximation and the so-called Gap-equation used e.g. in the Nambu-Jona-Lasinio model [1], which both give a first approximation for the many-body problem. An improvement is obtained by including correlations in the excited states as well as the ground state of the system. In nonrelativistic many-body theory this leads to the particle-hole Random-Phase-Approximation (RPA) equations.

The RPA equations have been derived by various methods (see e.g. ref. [2]). The most systematic approach, however, is the Green's function method, especially in view of possible generalizations. Although the connection of the RPA equations to the Bethe-Salpeter equation is mentioned in the literature [3], a clear and systematic description appears to be missing. It is the purpose of this paper to close this gap and to establish the relation to the Salpeter equation.

The paper is organized in the following way: In Sec.II we will give a derivation of the RPA equations based on the Green's function method. The derivation uses two approximations, i.e.

- we apply the instantaneous approximation to the amputated two-fermion propagator
- we substitute the full one-fermion propagators by the free ones.

In relativistic field theory the same approximations lead from the fermion-antifermion Bethe-Salpeter equation [4,5] to the Salpeter equation [6]. The structure of this equation shows many similarities with the RPA equations (see e.g. refs. [7,8]). We will show in Sec.III that Salpeter equation and RPA equations are indeed equivalent. Concluding remarks are given in Sec.IV.

II. RPA EQUATIONS AND THE INSTANTANEOUS APPROXIMATION

A. Pole structure of the polarization propagator

Let H be the hamiltonian describing the dynamics of a nonrelativistic or relativistic system of many interacting identical fermions with ground state $|\psi_0\rangle$ and excited states $|\psi_\lambda\rangle$, $\lambda > 0$. The corresponding energies will be denoted as E_0 and E_λ with $E_\lambda \leq E_{\lambda'}$ for $\lambda < \lambda'$. For simplicity we assume a discrete energy spectrum, i.e. free states are considered in a finite space volume. In the relativistic case $|\psi_0\rangle$ is usually the vacuum so that the excited states are particle-hole excitations of the vacuum, e.g. mesons.

In order to keep the discussion complete and to introduce the notation some well-known definitions and facts will be recalled in the following.

Let a_α, $a_\alpha^\dagger$ be fermion field operators with the anticommutator given by $\left\{a_\alpha^\dagger, a_\beta\right\} = \delta_{\alpha\beta}$. They correspond to an orthonormal single particle basis φ_α which will be specified later. The Heisenberg-picture will be used in the following and we define $A_\alpha(t) := e^{iHt}\, a_\alpha\, e^{-iHt}$. The two-fermion propagator in this basis is then given by

$$
\begin{aligned}
& i^2\, [G(t,t',u,u')]_{\alpha\alpha'\beta\beta'} = \\
& = \quad \langle\psi_0| T\, A_\alpha(t)\, A_{\alpha'}(t')\, A_\beta^\dagger(u)\, A_{\beta'}^\dagger(u')\, |\psi_0\rangle \\
& = -\langle\psi_0| T\, A_\beta^\dagger(u)\, A_\alpha(t)\, A_{\beta'}^\dagger(u')\, A_{\alpha'}(t')\, |\psi_0\rangle
\end{aligned}
\tag{1}
$$

(compare fig.1).

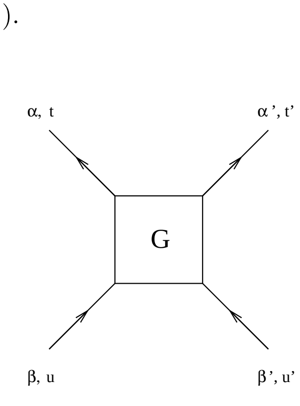

FIG. 1. The two-fermion propagator G

Let $u = t + \epsilon$ and $u' = t' + \epsilon$ with $\epsilon > 0$. In the limit $\epsilon \to 0$ one has

$$\begin{aligned}&\lim_{\epsilon\to 0}\,[G(t,t',t+\epsilon,t'+\epsilon)]_{\alpha\alpha'\beta\beta'} = \\ &= \Theta(t-t')\,\langle\psi_0|\,A^\dagger_\beta(t)\,A_\alpha(t)\,A^\dagger_{\beta'}(t')\,A_{\alpha'}(t')\,|\psi_0\rangle + \\ &+ \Theta(t'-t)\,\langle\psi_0|\,A^\dagger_{\beta'}(t')\,A_{\alpha'}(t')\,A^\dagger_\beta(t)\,A_\alpha(t)\,|\psi_0\rangle \qquad (2)\end{aligned}$$

and with $1 = \sum_\lambda |\psi_\lambda\rangle\langle\psi_\lambda|$ one obtains

$$\begin{aligned}&\lim_{\epsilon\to 0}\,[G(t,t',t+\epsilon,t'+\epsilon)]_{\alpha\alpha'\beta\beta'} = \\ &= \sum_\lambda \Big[\Theta(t-t')\,e^{-i(E_\lambda-E_0)\,(t-t')} \\ &\quad \langle\psi_0|a^\dagger_\beta a_\alpha|\psi_\lambda\rangle\,\langle\psi_\lambda|a^\dagger_{\beta'}a_{\alpha'}|\psi_0\rangle + \\ &\quad + \Theta(t'-t)\,e^{-i(E_\lambda-E_0)\,(t'-t)} \\ &\quad \langle\psi_0|a^\dagger_{\beta'}a_{\alpha'}|\psi_\lambda\rangle\,\langle\psi_\lambda|a^\dagger_\beta a_\alpha|\psi_0\rangle\Big] \qquad (3)\end{aligned}$$

The term for the ground state ($\lambda = 0$) can be rewritten in terms of one-fermion propagators $S^F_{\alpha\beta}(t-t') = -i\,\langle\psi_0|T\,A_\alpha(t)\,A^\dagger_\beta(t')|\psi_0\rangle$ as

$$\begin{aligned}&\langle\psi_0|a^\dagger_\beta a_\alpha|\psi_0\rangle\,\langle\psi_0|a^\dagger_{\beta'}a_{\alpha'}|\psi_0\rangle = \\ &= -\lim_{\epsilon\to 0}\,S^F_{\alpha\beta}(-\epsilon)\,S^F_{\alpha'\beta'}(-\epsilon) \qquad (4)\end{aligned}$$

The polarization propagator $\Pi(t-t')$ is now defined as

$$\begin{aligned}&\lim_{\epsilon\to 0}\,[G(t,t',t+\epsilon,t'+\epsilon)]_{\alpha\alpha'\beta\beta'} = \\ &= -\lim_{\epsilon\to 0}\,S^F_{\alpha\beta}(-\epsilon)\,S^F_{\alpha'\beta'}(-\epsilon) + i\,[\Pi(t-t')]_{\alpha\alpha'\beta\beta'} \qquad (5)\end{aligned}$$

so that

$$\begin{aligned}&i\,[\Pi(t-t')]_{\alpha\alpha'\beta\beta'} = \\ &= \sum_{\lambda\neq 0}\Big[\Theta(t-t')\,e^{-i(E_\lambda-E_0)\,(t-t')} \\ &\quad \langle\psi_0|a^\dagger_\beta a_\alpha|\psi_\lambda\rangle\,\langle\psi_\lambda|a^\dagger_{\beta'}a_{\alpha'}|\psi_0\rangle + \\ &\quad + \Theta(t'-t)\,e^{-i(E_\lambda-E_0)\,(t'-t)} \\ &\quad \langle\psi_0|a^\dagger_{\beta'}a_{\alpha'}|\psi_\lambda\rangle\,\langle\psi_\lambda|a^\dagger_\beta a_\alpha|\psi_0\rangle\Big] \qquad (6)\end{aligned}$$

With

$$\Theta(t)\,e^{-iEt} = \frac{1}{2\pi i}\int \frac{e^{-i\nu t}\,d\nu}{E-\nu-i\epsilon} \qquad (7)$$

one can write the fourier transform $\Pi(\nu) = \int dt\,e^{i\nu t}\,\Pi(t)$ as

$$\begin{aligned}&[\Pi(\nu)]_{\alpha\alpha'\beta\beta'} = \\ &= \sum_{\lambda\neq 0}\Bigg[\frac{\langle\psi_0|a^\dagger_\beta a_\alpha|\psi_\lambda\rangle\,\langle\psi_\lambda|a^\dagger_{\beta'}a_{\alpha'}|\psi_0\rangle}{\nu-(E_\lambda-E_0-i\epsilon)} \\ &\quad - \frac{\langle\psi_0|a^\dagger_{\beta'}a_{\alpha'}|\psi_\lambda\rangle\,\langle\psi_\lambda|a^\dagger_\beta a_\alpha|\psi_0\rangle}{\nu+(E_\lambda-E_0-i\epsilon)}\Bigg] \qquad (8)\end{aligned}$$

Thus the energy levels of the excited states appear as a doubled system of poles in the polarization propagator $\Pi(\nu)$ at $\nu = \pm(E_\lambda - E_0 - i\epsilon)$ as shown in fig.2.

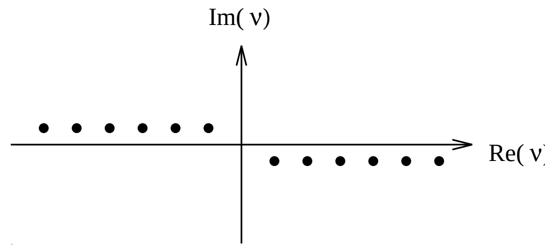

FIG. 2. Location of the poles of $\Pi(\nu)$

This doubling of the spectrum can be traced back to the appearence of the time ordering operator T in the definition of the two-fermion propagator G. Any equation that looks for the poles of Π will therefore obtain a doubled eigenvalue spectrum. This statement holds for relativistic as well as nonrelativistic calculations. It is clear that the appearence of the second set of poles does not yield any further physical information, since only half of the poles can be identified with the eigenvalues of the hamiltonian H.

B. The instantaneous approximation

The amputated two-fermion propagator M is defined as (see also fig.3)

$$\begin{aligned}&[G(t,t',u,u')]_{\alpha\alpha'\beta\beta'} = \\ &-S^F_{\alpha\beta}(t-u)\,S^F_{\alpha'\beta'}(t'-u') + S^F_{\alpha\beta'}(t-u')\,S^F_{\alpha'\beta}(t'-u) + \\ &+\sum_{\alpha_1\alpha_2}\sum_{\alpha_3\alpha_4}\int dt_1dt_2dt_3dt_4\,S^F_{\alpha\alpha_1}(t-t_1)\,S^F_{\alpha'\alpha_2}(t'-t_2)\,\cdot \\ &\cdot\,[M(t_1,t_2,t_3,t_4)]_{\alpha_1\alpha_2\alpha_3\alpha_4}\,S^F_{\alpha_3\beta}(t_3-u)\,S^F_{\alpha_4\beta'}(t_4-u') \quad (9)\end{aligned}$$

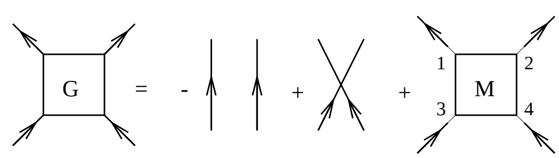

FIG. 3. Definition of the amputated two-fermion propagator M

The instantaneous approximation of M used for the investigation of one-particle – one-hole propagation is given by the ansatz

$$M \;\to\; M^{inst} = \delta(t_1-t_3)\,\delta(t_2-t_4)\,\Gamma(t_1-t_2) \qquad (10)$$

i.e. it is assumed that particles and holes interact instantaneously with each other.

In nonrelativistic many-body theory this ansatz is equivalent to taking into account only diagrams that have the appropriate instantaneous structure, as shown in fig.4.

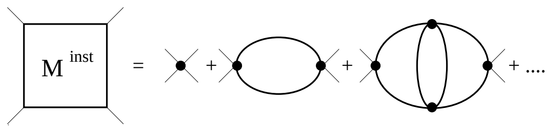

FIG. 4. Diagrams that contribute to M^{inst} for a 4-point-interaction

The same statement also holds for a relativistic 4-point-interaction. In more general relativistic field theories like QCD, however, the instantaneous approximation cannot easily be interpreted in terms of Feynman diagrams.

With the relations of the previous section (using $t' = 0$) the instantaneous approximation yields for the polarization propagator

$$\begin{aligned} i\,[\Pi^{inst}(t)]_{\alpha\alpha'\beta\beta'} &= S^F_{\alpha\beta'}(t)\,S^F_{\alpha'\beta}(-t)+ \\ &+\sum_{\alpha_1\alpha_2}\sum_{\alpha_3\alpha_4}\int dt_1 dt_2\, S^F_{\alpha\alpha_1}(t-t_1)\,S^F_{\alpha'\alpha_2}(-t_2)\cdot \\ &\cdot[\Gamma(t_1-t_2)]_{\alpha_1\alpha_2\alpha_3\alpha_4}\,S^F_{\alpha_3\beta}(t_1-t)\,S^F_{\alpha_4\beta'}(t_2) \end{aligned} \quad (11)$$

To simplify the notation we define

$$[g(t)]_{\alpha\alpha'\beta\beta'} := S^F_{\alpha\beta'}(t)\,S^F_{\alpha'\beta}(-t) \quad (12)$$

Furthermore let

$$(A\,B)_{\alpha\beta\gamma\delta} := \sum_{\alpha'\beta'} A_{\alpha\alpha'\gamma\beta'}\,B_{\beta'\beta\alpha'\delta} \quad (13)$$

An easy way to represent this definition is to define multi-indices as $A_{\alpha\beta\gamma\delta} =: A_{(\alpha\gamma)\,(\delta\beta)} =: A_{ij}$ so that the usual matrix multiplication can be applied to A_{ij}. With these definitions the Fourier transform of eq.(11) can be written as

$$i\,\Pi^{inst}(\nu) = g(\nu) + g(\nu)\,\Gamma(\nu)\,g(\nu) \quad (14)$$

The exact Bethe-Salpeter equation for the amputated two-fermion propagator M reads

$$\begin{aligned} M_{1234} = K_{1234} + \sum_{\alpha_5\alpha_6}\sum_{\alpha_7\alpha_8}\int dt_5\,dt_6\,dt_7\,dt_8 \\ \left[K_{1537}\,S^F_{58}\,S^F_{67}\,M_{6284} + K_{2537}\,S^F_{58}\,S^F_{67}\,M_{6184}\right] \end{aligned} \quad (15)$$

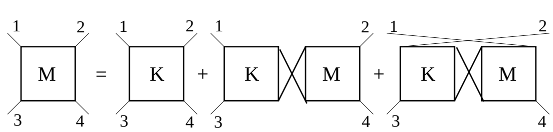

FIG. 5. The Bethe-Salpeter equation for the amputated two-fermion propagator M

(compare fig.5) with an appropriate particle-hole kernel K and the notation $[K(t_a,t_b,t_c,t_d)]_{\alpha_a\alpha_b\alpha_c\alpha_d} =: K_{abcd}$ and analogously for M and S^F.

In a first step one neglects the last (exchange) term. The second step is to substitute M^{inst} for M which implies that also K must have this instantaneous structure, i.e.

$$K \;\rightarrow\; K^{inst} = \delta(t_1-t_3)\,\delta(t_2-t_4)\,iV(t_1-t_2) \quad (16)$$

After applying the Fourier transformation the instantaneous Bethe-Salpeter equation reads

$$\Gamma(\nu) = iV(\nu) + iV(\nu)\,g(\nu)\,\Gamma(\nu) \quad (17)$$

Together with $i\,\Pi^{inst} = g + g\,\Gamma\,g$ one obtains $i\,\Pi^{inst} = g - g\,V\,\Pi^{inst}$ or

$$i\,\Pi^{inst}(\nu) = \left([g(\nu)]^{-1} - i\,V(\nu)\right)^{-1} \quad (18)$$

C. The RPA-equations

From eq.(8) we know that $\Pi(\nu)$ has poles at $\nu = \pm(E_\lambda - E_0 - i\epsilon)$. On the other hand we can use eq.(18) to obtain a spectral decomposition for $\Pi^{inst}(\nu)$ in order to calculate the pole positions and matrix elements in the instantaneous approximation. In the following we therefore look for solutions $F(\nu)$ of the equation

$$\left([g(\nu)]^{-1} - i\,V(\nu)\right)\,F(\nu) = 0 \quad (19)$$

To proceed further we approximate the full propagators S^F in g by the free propagators $[S^F_0(t)]_{\alpha\beta} = -i\,\langle\chi_0|T\,A^0_\alpha(t)\,[A^0_\beta(0)]^\dagger|\chi_0\rangle$ with $A^0_\alpha(t) = e^{iH_0t}\,a_\alpha\,e^{-iH_0t}$, where $H_0 = \sum_\alpha \epsilon_\alpha\,a^\dagger_\alpha a_\alpha$ is some 'free' hamiltonian with ground state $|\chi_0\rangle$. The basis states φ_α have thus been chosen as eigenstates of H_0, i.e. $H_0\,\varphi_\alpha = \epsilon_\alpha\,\varphi_\alpha$. In the nonrelativistic case an appropriate choice for H_0 is the Hartree-Fock hamiltonian, whereas in the relativistic case one can use the free Dirac hamiltonian with some effective fermion mass m.

For a system of n fermions, the ground state $|\chi_0\rangle$ of H_0 is a product wavefunction where the lowest n eigenstates are occupied, i.e. $a_\alpha\,|\chi_0\rangle = 0$ for $\alpha > n$ and $a^\dagger_\alpha\,|\chi_0\rangle = 0$ for $\alpha \le n$. In the relativistic case $|\chi_0\rangle$ is the filled Dirac sea. With

$$H_0\,a^\dagger_\alpha\,|\chi_0\rangle = (\epsilon_\alpha + \epsilon_0)\,a^\dagger_\alpha\,|\chi_0\rangle \quad (20)$$

$$H_0\,a_\alpha\,|\chi_0\rangle = (-\epsilon_\alpha + \epsilon_0)\,a_\alpha\,|\chi_0\rangle \quad (21)$$

one finds

$$\begin{aligned} [iS^F_0(t)]_{\alpha\beta} &= e^{-i\epsilon_\alpha t}\,\delta_{\alpha\beta} \\ &\cdot[\Theta(t)\,\Theta(\alpha-n) - \Theta(-t)\,\Theta(n-\alpha)] \end{aligned} \quad (22)$$

and therefore

$$[g^0(t)]_{\alpha\alpha'\beta\beta'} = [S_0^F(t)]_{\alpha\beta'}\,[S_0^F(-t)]_{\alpha'\beta} = \\ = -e^{-(\epsilon_\alpha-\epsilon_\beta)t}\,\delta_{\alpha\beta'}\,\delta_{\alpha'\beta}\;\cdot \\ \cdot\big[\Theta(t)\,\Theta(\alpha-n)\,\Theta(n-\beta)+ \\ +\Theta(-t)\,\Theta(\beta-n)\,\Theta(n-\alpha)\big] \qquad (23)$$

In the following we will use capital letters like $\alpha = A$ for $\alpha > n$ and small letters like $\alpha = a$ for $\alpha \le n$. Using eq.(7) for the Θ-functions and the multi-index notation of the previous section we find for the fourier transform of g^0

$$\left[\left(g^0(\nu)\right)^{-1}\right]_{(Ab)\,(B'a')} = -i\,(\epsilon_A-\epsilon_b-\nu)\,\delta_{(Ab)\,(B'a')} \qquad (24)$$

$$\left[\left(g^0(\nu)\right)^{-1}\right]_{(aB)\,(b'A')} = -i\,(\epsilon_B-\epsilon_a+\nu)\,\delta_{(aB)\,(b'A')} \qquad (25)$$

Note that the matrix elements of g are zero if one of the multi-indices equals (ab) or (AB). Because of $i\Pi^{inst} = g + g\Gamma g$ the same holds for Π^{inst} and because of $i\Pi^{inst} = g - gV\Pi^{inst}$ also the matrix elements of V with these indices are not relevant. It is therefore sufficient to consider only the index combinations (Ab) and (bA). This simplification is a consequence of approximating the full fermion propagators by the free ones. Write

$$F = \begin{pmatrix} X \\ Y \end{pmatrix} \qquad (26)$$

with $X_k = F_{(Ab)}$ and $Y_k = F_{(bA)}$. In the following we will assume that X, Y are vectors of dimension D, i.e. $k = 1 \ldots D$. Define $2D \times 2D$ -matrices Ω and N by $[g^0(\nu)]^{-1} =: -i\,(\Omega - \nu\,N)$ so that

$$\Omega_{(Ab)(B'a')} = (\epsilon_A-\epsilon_b)\,\delta_{(Ab)(B'a')} \\ \Omega_{(aB)(b'A')} = (\epsilon_B-\epsilon_a)\,\delta_{(aB)(b'A')} \\ N_{(Ab)(B'a')} = \delta_{(Ab)(B'a')} \\ N_{(aB)(b'A')} = -\delta_{(aB)(b'A')} \qquad (27)$$

which we will write in a simplified matrix notation as

$$\Omega = \begin{pmatrix} (\epsilon_A-\epsilon_b) & 0 \\ 0 & (\epsilon_B-\epsilon_a) \end{pmatrix} \quad , \quad N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (28)$$

Setting $h := \Omega + V$ we find that eq.(19) (multiplied with i) can be written as

$$h\,F^\rho = \nu_\rho\,N\,F^\rho \qquad (29)$$

where ρ labels the different solutions and $F^\rho := F(\nu_\rho)$. In the nonrelativistic case one usually approximates the kernel V by its lowest order contribution, i.e. the two-fermion potential. Then this equation is exactly the RPA-equation of e.g. ref. [2].

Up to now it is not clear how to connect ν_ρ and F^ρ with the eigenvalues and eigenstates of the full Hamiltonian H. It is usefull at this point to recall some of the properties of the RPA-equation:

- h is a hermitian matrix and has the structure

$$h = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \qquad (30)$$

- Let

$$F^\sigma = \begin{pmatrix} X^\sigma \\ Y^\sigma \end{pmatrix} \qquad (31)$$

  be a solution with eigenvalue ν_σ. Then

$$F^\tau = \begin{pmatrix} X^\tau \\ Y^\tau \end{pmatrix} = \begin{pmatrix} (Y^\sigma)^* \\ (X^\sigma)^* \end{pmatrix} \qquad (32)$$

  is a solution with eigenvalue $\nu_\tau = -\nu_\sigma$ (we assume that ν_σ is real). For the components of the eigenvectors this means that $F^\tau_{(Ab)} = [F^\sigma_{(bA)}]^*$ and $F^\tau_{(bA)} = [F^\sigma_{(Ab)}]^*$.

- If F^{ρ_1} and F^{ρ_2} are solutions with $\nu_{\rho_1} \ne \nu_{\rho_2}$ then

$$\langle F^{\rho_1}|N|\,F^{\rho_2}\rangle := (F^{\rho_1})^\dagger\,N\,F^{\rho_2} = 0 \qquad (33)$$

- If B is 'small enough' the $2D$ eigenvalues ν_ρ are real and nonzero. We will use the index convention $\sigma = 1\;\ldots\;D$ for $\nu_\sigma > 0$ and $\tau = D+1\;\ldots\;2D$ for $\nu_\tau < 0$ with $\nu_\tau = -\nu_\sigma$. The solutions can then be normalized as

$$\langle F^{\rho'}|N|\,F^\rho\rangle = N_\rho\,\delta_{\rho'\rho} \qquad (34)$$

  with $N_\sigma = 1$ and $N_\tau = -1$. They form a $2D$-dimensional basis, i.e. one can expand a vector F as $F = \sum_\rho c_\rho\,F^\rho$ with $c_\rho = N_\rho^{-1}\,\langle F^\rho|N|\,F\rangle$ so that

$$1 = \sum_\rho N_\rho^{-1}\,|F^\rho\rangle\,\langle F^\rho|N| \qquad (35)$$

The following calculation

$$(N\nu-h)^{-1}\,N|F^{\rho'}\rangle = \\ = (\nu-N^{-1}h)^{-1}\,|F^{\rho'}\rangle = \frac{1}{\nu-\nu_{\rho'}}\,|F^{\rho'}\rangle = \\ = \sum_\rho \frac{1}{\nu-\nu_\rho}\,N_\rho^{-1}\,|F^\rho\rangle\,\langle F^\rho|N|F^{\rho'}\rangle \qquad (36)$$

now showes that the spectral decomposition of $(N\nu - h)^{-1}$ is given by

$$(N\nu-h)^{-1} = \sum_{\rho=1}^{2D} \frac{1}{\nu-\nu_\rho}\,N_\rho^{-1}\,|F^\rho\rangle\,\langle F^\rho| \qquad (37)$$

Let $(\alpha\beta)$ stand for (Ab) or (aB). Since $F^\tau_{(\alpha\beta)} = [F^\sigma_{(\beta\alpha)}]^*$ the spectral decomposition for the matrix elements of $\Pi^{inst}(\nu)$ can be written as

$$[\Pi^{inst}(\nu)]_{\alpha\delta\beta\gamma} = (N\nu - h)^{-1}_{(\alpha\beta)\,(\gamma\delta)} =$$

$$= \sum_{\sigma=1}^{M} \left[\frac{1}{\nu - \nu_\sigma} F^\sigma_{(\alpha\beta)} \, [F^\sigma_{(\gamma\delta)}]^* \; - \frac{1}{\nu + \nu_\sigma} [F^\sigma_{(\beta\alpha)}]^* \, F^\sigma_{(\delta\gamma)} \right] \quad (38)$$

Comparing this equation with the exact pole structure given in eq.(8) we can identify

$$F^\sigma_{\alpha\beta} = \langle \psi_0 | a^\dagger_\beta a_\alpha | \psi_\sigma \rangle \quad (39)$$

$$\nu_\sigma = E_\sigma - E_0 \quad (40)$$

III. FROM THE SALPETER EQUATION TO THE RPA-EQUATIONS

In relativistic field theory particle-hole excitations of the fermionic vacuum are described by the fermion-antifermion Bethe-Salpeter equation [4,5]. In the instantaneous approximation this equation reduces to the Salpeter equation [6]. From the considerations of the previous section we expect the Salpeter equation to be equivalent to the RPA equations (29). We will show in this section that this is indeed the case.

The Salpeter equation for one fermion flavor can be written in the form (see e.g. [7,8])

$$(\mathcal{H}\psi)(\vec{p}) = M\,\psi(\vec{p}) \quad (41)$$

where

$$\begin{aligned}(\mathcal{H}\psi)(\vec{p}) &= H(\vec{p})\psi(\vec{p}) - \psi(\vec{p})H(\vec{p}) \\ &- \int \frac{d^3p'}{(2\pi)^3} \Lambda^+(\vec{p})\,[W(\vec{p},\vec{p}\,')\,\psi(\vec{p}\,')]\,\Lambda^-(\vec{p}) \\ &+ \int \frac{d^3p'}{(2\pi)^3} \Lambda^-(\vec{p})\,[W(\vec{p},\vec{p}\,')\,\psi(\vec{p}\,')]\,\Lambda^+(\vec{p}) \end{aligned} \quad (42)$$

with the free Dirac hamiltonian $H(\vec{p}) = \gamma^0(\vec{\gamma}\vec{p} + m)$, the projection operators $\Lambda^\pm = (\omega \pm H(\vec{p}))/(2\omega)$ and $\omega = \sqrt{\vec{p}^{\,2} + m^2}$ (don't confuse $H(\vec{p})$ with the full hamiltonian of the previous section). Here M is the mass of the bound state, m is the effective fermion mass and W is the instantaneous interaction kernel. One can define a scalar product by

$$\langle \psi_1 \,|\psi_2 \rangle = \int \mathrm{tr} \left(\psi_1^\dagger \, \Lambda^+ \, \psi_2 \, \Lambda^- - \psi_1^\dagger \, \Lambda^- \, \psi_2 \, \Lambda^+ \right) \quad (43)$$

with all quantities depending on $\vec{p}$ and the notation $\int = \int d^3p/(2\pi)^3$.

Let $u(\vec{p})$, $v(\vec{p})$ be free Dirac spinors (we use the conventions of [9] in the following). Define

$$\begin{aligned} w^{(+)}_{rs}(\vec{p}) &:= u_r(\vec{p}) \otimes v^\dagger_s(-\vec{p}) \\ w^{(-)}_{rs}(\vec{p}) &:= v_s(-\vec{p}) \otimes u^\dagger_r(\vec{p}) \end{aligned} \quad (44)$$

We will use the box normalization in the following, i.e. we substitute

$$\int \frac{d^3p}{(2\pi)^3} \;\longrightarrow\; \frac{1}{V} \sum_{\vec{p}} \quad (45)$$

Since $\Lambda^+\psi\,\Lambda^+ = \Lambda^-\psi\,\Lambda^- = 0$ we can expand

$$\psi(\vec{p}) = \sqrt{V}\,\frac{m}{\omega} \sum_{r,s=\pm 1/2} \left(b^{(+)}_{\vec{p},rs}\, w^{(+)}_{rs}(\vec{p}) + b^{(-)}_{\vec{p},rs}\, w^{(-)}_{rs}(\vec{p}) \right) \quad (46)$$

with some suitable coefficients b. The factors in front of the summation sign have been choosen to simplify the notation in the following.

Solving the Salpeter equation $\mathcal{H}\psi = M\,\psi$ is equivalent to solving

$$\langle \psi_1 | \mathcal{H}\,\psi_2 \rangle = M\,\langle \psi_1 | \psi_2 \rangle \quad (47)$$

for all given ψ_1.

With eq.(46) and the relations for u, v of ref. [9] we compute

$$\mathrm{tr}\left[[w^{(+)}_{rs}]^\dagger\, \Lambda^+\, w^{(+)}_{r's'}\, \Lambda^- - [w^{(+)}_{rs}]^\dagger\, \Lambda^-\, w^{(+)}_{r's'}\, \Lambda^+ \right] = = +\frac{\omega^2}{m^2}\,\delta_{rr'}\,\delta_{ss'} \quad (48)$$

$$\mathrm{tr}\left[[w^{(-)}_{rs}]^\dagger\, \Lambda^+\, w^{(-)}_{r's'}\, \Lambda^- - [w^{(-)}_{rs}]^\dagger\, \Lambda^-\, w^{(-)}_{r's'}\, \Lambda^+ \right] = = -\frac{\omega^2}{m^2}\,\delta_{rr'}\,\delta_{ss'} \quad (49)$$

$$\mathrm{tr}\left[[w^{(+)}_{rs}]^\dagger\, \Lambda^+\, w^{(-)}_{r's'}\, \Lambda^- - [w^{(+)}_{rs}]^\dagger\, \Lambda^-\, w^{(-)}_{r's'}\, \Lambda^+ \right] = = 0 \quad (50)$$

$$\mathrm{tr}\left[[w^{(-)}_{rs}]^\dagger\, \Lambda^+\, w^{(+)}_{r's'}\, \Lambda^- - [w^{(-)}_{rs}]^\dagger\, \Lambda^-\, w^{(+)}_{r's'}\, \Lambda^+ \right] = = 0 \quad (51)$$

where all quantities depend on $\vec{p}$. Using the multiindex $i = (\vec{p}, r, s)$ the scalar product in the box normalization can therefore be written as

$$\begin{aligned}\langle \psi_1 | \psi_2 \rangle &= \sum_i \left[(b_1^{(+)})^*_i\, (b_2^{(+)})_i - (b_1^{(-)})^*_i\, (b_2^{(-)})_i \right] = \\ &= \begin{pmatrix} b_1^{(+)} \\ b_1^{(-)} \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} b_2^{(+)} \\ b_2^{(-)} \end{pmatrix} \end{aligned} \quad (52)$$

We further have

$$\langle \psi_1 | \mathcal{H}\,\psi_2 \rangle = \langle \psi_1 | \mathcal{T}\,\psi_2 \rangle + \langle \psi_1 | \mathcal{V}\,\psi_2 \rangle \quad \text{with} \quad (53)$$

$$\langle \psi_1 | \mathcal{T}\,\psi_2 \rangle = \frac{1}{V} \sum_{\vec{p}} 2\omega\, \mathrm{tr}\left(\psi_1^\dagger(\vec{p})\,\psi_2(\vec{p}) \right) \quad (54)$$

$$\langle \psi_1 | \mathcal{V}\,\psi_2 \rangle = -\frac{1}{V^2} \sum_{\vec{p}} \sum_{\vec{p}\,'} \mathrm{tr}\left(\psi_1^\dagger(\vec{p})\, W(\vec{p},\vec{p}\,')\, \psi_2(\vec{p}\,') \right) \quad (55)$$

We proceed analogously for the kinetic energy term and compute

$$\mathrm{tr}\,\left[[w_{rs}^{(a)}(\vec{p})]^\dagger\,[w_{r's'}^{(a')}(\vec{p})]\right] = \frac{\omega^2}{m^2}\,\delta_{rr'}\,\delta_{ss'}\,\delta_{aa'} \qquad (56)$$

(where $a, a' = \pm$) so that with $\omega_i = \omega(\vec{p})$

$$\langle\psi_1|\mathcal{T}\,\psi_2\rangle = \sum_i 2\omega_i\,\left[(b_1^{(+)})_i^*\,(b_2^{(+)})_i + (b_1^{(-)})_i^*\,(b_2^{(-)})_i\right] = \\ =: \begin{pmatrix} b_1^{(+)} \\ b_1^{(-)} \end{pmatrix}^\dagger \begin{pmatrix} 2\omega & 0 \\ 0 & 2\omega \end{pmatrix} \begin{pmatrix} b_2^{(+)} \\ b_2^{(-)} \end{pmatrix} \qquad (57)$$

For the interaction term we define

$$V_{ij}^{a_1 a_2} := -\frac{1}{V}\,\frac{m}{\omega}\,\frac{m}{\omega'}\,\mathrm{tr}\,\left([w_{r_1 s_1}^{(a_1)}(\vec{p})]^\dagger\,W(\vec{p},\vec{p}\,')\,w_{r_2 s_2}^{(a_2)}(\vec{p}\,')\right) \qquad (58)$$

with $a_1, a_2 = \pm$ and the multiindices $i = (\vec{p}, r_1, s_1)$, $j = (\vec{p}\,', r_2, s_2)$. Consider interaction kernels that fulfill the relation $[W(\vec{p},\vec{p}\,')\,\psi(\vec{p}\,')]^\dagger = W(\vec{p},\vec{p}\,')\,[\psi(\vec{p}\,')]^\dagger$ (this is usually the case for kernels of physical interest). Since $[w_{rs}^{(+)}(\vec{p})]^\dagger = w_{rs}^{(-)}(\vec{p})$ we have $V_{ij}^{--} = (V_{ij}^{++})^*$ and $V_{ij}^{-+} = (V_{ij}^{+-})^*$ so that we can write

$$\langle\psi_1|\mathcal{V}\,\psi_2\rangle = \sum_{i,j}\sum_{a_1,a_2=\pm}(b_1^{(a_1)})_i^*\,V_{ij}^{a_1 a_2}\,(b_2^{(a_2)})_j \\ = \begin{pmatrix} b_1^{(+)} \\ b_1^{(-)} \end{pmatrix}^\dagger \begin{pmatrix} V^{++} & V^{+-} \\ (V^{+-})^* & (V^{++})^* \end{pmatrix} \begin{pmatrix} b_2^{(+)} \\ b_2^{(-)} \end{pmatrix} \qquad (59)$$

Since $b_1^{(\pm)}$ are arbitrary the Salpeter equation can now be written in matrix form as

$$\left[\begin{pmatrix} 2\omega & 0 \\ 0 & 2\omega \end{pmatrix} + \begin{pmatrix} V^{++} & V^{+-} \\ (V^{+-})^* & (V^{++})^* \end{pmatrix}\right] \begin{pmatrix} b^{(+)} \\ b^{(-)} \end{pmatrix} = \\ = M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} b^{(+)} \\ b^{(-)} \end{pmatrix} \qquad (60)$$

We identify the eigenvalues of the free hamiltonian with the kinetic energies of the free fermions as $\epsilon_A = -\epsilon_a = \omega_i$ and $\epsilon_B = -\epsilon_b = \omega_i$. Further we identify the positive eigenvalues M with $\nu_\sigma = E_\sigma - E_0$ and

$$\begin{pmatrix} b^{(+)} \\ b^{(-)} \end{pmatrix} = F^\sigma = \begin{pmatrix} X^\sigma \\ Y^\sigma \end{pmatrix} \qquad (61)$$

Therefore we find that the Salpeter equation (60) has exactly the form of the RPA-equations (29).

In this context we would like to mention a work of J.Piekarewicz [10] which gives a direct derivation of the Salpeter equation using a method similar to our derivation of the RPA equations as given in section II.

IV. CONCLUSION

In the present paper we have shown that the RPA equations can be derived by applying the instantaneous approximation to the amputated two-fermion propagator and by approximating the full fermion propagators by the free ones. Our derivation holds for nonrelativistic as well as relativistic fermionic systems. Since in relativistic field theory the same approximations lead from the fermion-antifermion Bethe-Salpeter equation to the Salpeter equation, this equation should be equivalent to the RPA equations. We have shown explicitely that this is indeed the case.

The RPA equations have been analyzed carefully by many authors, especially in the context of nuclear physics (compare e.g. the references given in [2]). It would be interesting to transfer these results to the Salpeter equation, e.g. results on the stability of the RPA or on the appearence of spurious solutions.

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